I+

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Calculus 252
Demonstration Problem
Level I

Excellent presentation. A couple of technical points. 1. Didn't make it clear we're getting upper (circumscribing) sums. 2. It would be better (clearer to reader) to have graph closer to verbal descriptions.

Demonstration Problem

Level I

We are very well versed with finding the areas of shapes such as squares, triangles, rectangles, circles, etc. But what if we have to find the area of a curve?

Let's investigate this function: f(x)=x^2

We are going to figure out how to determine the area underneath this curve from x=0 to x=a.

There is no straight out way to find the area of a curve. However, we can still accomplish this task by making many rectangles under the curve because we know how to find the areas of rectangles. After we compute the areas of these rectangles, we will add up those values and get a good estimate of the area underneath the curve. The main idea is that the more rectangles we make, the more accurate our area approximation will become.

The rectangles that we will be making will have a certain uniform width, and will also touch the x^2 curve at a particular point. Specifically, we will be using upper (right) estimates, meaning that the right hand corner of our rectangles will touch the curve that x^2 makes.

We will first need to find the width of the rectangle that we will be using. The width of the rectangle, denoted as Δx, is the distance between our starting point and our ending point, divided by the number of rectangles we will be using. Mathematically this is expressed as:

$\Delta x = (b-a)/n$

Where a is the starting point, and b is the ending point. Is our case, our starting point is 0, and our ending point is a. So we determine our width like so:

Width of generic interval:
$\Delta x=$
$(b-a)/n=$
$(a-0)/n=$
a/n

The next thing we have to do is determine which function values to use. Remember that the area of a rectangle is base*height. Our base is our width, which is a/n. Our function value is whatever f(x) value occurs at our endpoints. So first of all, before we can determine the function values, we need to find out endpoints.

Our first endpoint is 0, since this is where our first point is. Every endpoint after this is a multiple of a/n, our width. This makes sense because every time we chose another endpoint, we are going over to the right another width's worth, or adding a/n every time. Here are our endpoints:

Right end-points:
$x_0=0$
$x_1=a/n$
$x_2=2a/n$
$x_3=3a/n$
$x_i=ia/n$

As you can see, for the generic endpoint x_i, the formula is ia/n.

Well, now we have the 3 things necessary in calculating area:

1. Width
2. Endpoints
3. Function values that we can derive from the endpoints.

Well said

Notice that we don't have any actual function values written down. This is because we are going from x=0 to x=a. However, if we were going from x=0 to x=3 say, then we'd have actual numbers to work with.

Now all that is left is to calculate the actual area underneath the curve of x^2 from x=0 to x=a.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$$

In this equation we see that we have our $f(x_i)$, which is our general function value of our endpoint, and this is multiplied by Δx, which is our width. These two values multiplied together produce the area of one rectangle. We also note that n is growing positively without bound. This means that the number of rectangles gets larger and larger, which gives us a more accurate area. And the sigma denotes that we are adding up all of the areas of the rectangles calculated to give us the total area under the curve.

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(ia/n)\ a/n$$

If we remember from above that our general endpoint is x_i, then we can substitute in ia/n for x_i.

$$\lim_{n\to\infty} a/n \sum_{i=1}^{n} (ia/n)\text{^}2$$

All we did here was move the a/n out to the left, since it is a multiplier.

$$\lim_{n\to\infty} a/n \sum_{i=1}^{n} (a\text{^}2*i\text{^}2)/n\text{^}2$$

Here some algebra was used to multiply our (ia/n)^2.

$$\lim_{n\to\infty} a/n*\ a\text{^}2 \sum_{i=1}^{n} i\text{^}2/n\text{^}2$$

Then here we moved out another multiplier, a^2.

• $\lim n\text{->}\infty \ a^3/n \sum_{i=1}^{n} i^2/n^2$

In this step we simply multiplied out $a/n^* \ a^2$.

• $\lim n\text{->}\infty \ a^3/n * ((n+1)(2n+1))/6n$

$((n+1)(2n+1))/6n$ was obtained by:

$$(\lim n\text{->}\infty \sum_{i=1}^{n} i^2) / n^2$$

This is because:

$$\sum_{i=1}^{n} i^2 = (n(n+1)(2n+1))/6$$

This is $2n^3+3n^2+n$. Now we have to divide this by 6, and we get:
$n^3/3 + n^2/2 + n/6$.

The next step is to divide $n^3/3 + n^2/2 + n/6$ by n^2, as shown above.
Doing this procedure we get:
$(2n^2+3n+1)/6n$ as noted about 15 lines up.

• $\lim n\text{->}\infty \ (a^3/n) * (2n^2+3n+1)/6n$

In this step we multiplied out $(n+1)(2n+1)$.

• $\lim n\text{->}\infty \ a^3/2n + a^3/6n^2 + a^3/3$

• $\lim n\text{->}\infty \ a^3 \ (1/2n + 1/6n^2 + 1/3)$

Here we factored out an a^3 from each term. We did this because it will help us determine the final limit, and that it is the highest term that each has in it.

•a^3 (lim n->∞ 1/2n + lim n->∞ 1/6n^2 + lim n->∞ 1/3)
In this step we brought out the a^3, and also separated a^3 * lim n->∞ (1/2n + 1/6n^2 + 1/3) into 3 different limits.

•a^3 (0+0+1/3)
Now we just evaluate these limits and find that all of them, except 1/3, approach 0 as n grows positively without bound. As a general informal rule, 1/∞=0.

•a^3 * 1/3
Now we simplify further by doing some elementary addition.

•a^3/3
And further simplify by realizing that a^3 * 1/3 is the same as a^3/3.

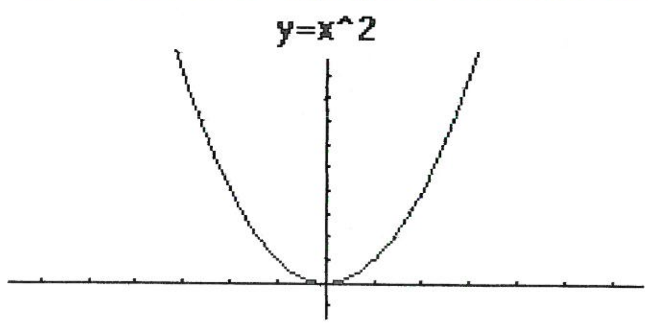
From this investigation we have concluded that:

$$\int_0^a x^2\, dx = a^3/3,$$

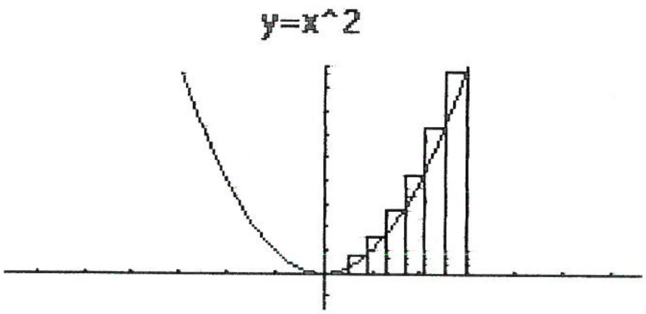
In other words, for x=0 to x=a, we can find the area under the curve of x^2 by substituting a into this formula: a^3/3.

Here is a pictorial view of the general process we went through. Each tick mark represents 1 unit.

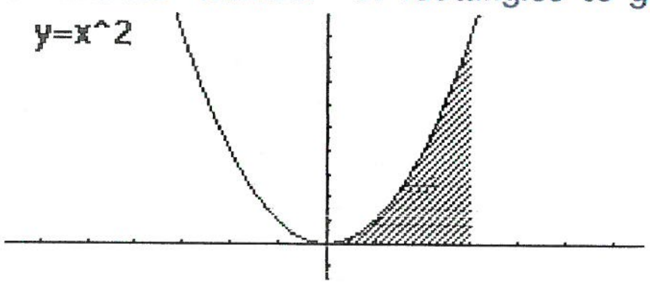
We started with a graph of our function x^2:



Then we made some rectangles on our graph to approximate our area:



Then, in essence, by taking the limit as n grew positively large, we made an infinite "number" of rectangles to get an exact value of the area:



These pictures are an oversimplified version of the actual process, but if we kept making more rectangles, our graph would start to look shaded in because the rectangles would continue to be more densely packed together.

In the end, we calculate all the areas of the rectangles, add them together, and we get a value for the area under the curve x^2, from 0 to a.

Below is a picture of a generic rectangle, with all necessary pieces of information labelled.

$f(x)=x^2$

Generic View